TD 13 – Chaînes de Markov (random walks)

Exercice 1. *Diffusion*

We study a simple model for the exchange of gas molecules between two containers. The total number of molecules in the two containers is N, and we study the evolution of the number of molecules in the first container. At every discrete time step t, the exchange is modeled as follows : if the first container has x molecules, then it increases to $x+1$ with probability $\frac{N-x}{N}$ and decreases to $x-1$ with probability $\frac{x}{N}$.

1. Describe this model with a Markov chain (give the state space and transition matrix). For $N = 3$, draw a graphical representation of this Markov chain.
2. Find the stationary distribution of this Markov chain.
3. Suppose at time 0, the first container is empty (i.e., all the N molecules are in the second container). Let $T \geq 1$ be the next time where the first container is empty. Compute $\mathbf{E}[T]$.

Exercice 2. *Fumeur*

Un fumeur décide d'arrêter de fumer. Le premier jour suivant cette bonne résolution (jour 0), il ne fume pas. On suppose que la probabilité qu'il fume le jour $j+1$ s'il n'a pas fumé le jour j est α, et que la probabilité qu'il ne fume pas le jour $j+1$ s'il a fumé le jour j est β, avec α et β non nuls et indépendants de j.

1. Justifier que l'on peut modéliser ce problème par une chaîne de Markov, et en donner sa représentation graphique.
2. Calculer la probabilité p_n qu'il ne fume pas le jour n. Quelle est la limite π de $(p_n, 1-p_n)$ quand $n \to +\infty$? Vérifier que π est une probabilité invariante pour la chaîne, c'est-à-dire que si X_n suit la loi π, alors X_{n+1} aussi.
3. Trouver $s > 0$ et $0 < t < 1$ tels que , pour tout état x on a : $|\mathbf{P}\{X_n = x\} - \pi(x)| \leq st^n$.
4. Quelle est la loi du premier jour où il se remet à fumer ?
5. Quelle est la loi du premier jour (autre que le jour 0) où il ne fume pas ?
6. Calculer l'espérance du nombre de jours N_n où il fume entre le jour 1 et le jour n. Déterminer la limite $\mathbf{E}[N_n]/n$.

Exercice 3. *Cover time in graphs*

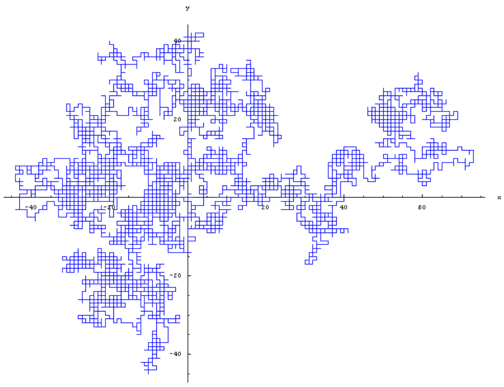
Given a finite, undirected non-bipartite and connected graph $G = (V, E)$, recall that the *cover time* of G is the maximum over all vertices $v \in V$ of the expected time to visit all of the nodes in the graph by a random walk starting from v.

1. Recall that $h_{v,u}$ is the expected number of steps to reach u from v and $h_{u,u} = \frac{2|E|}{d(u)}$. Show that
$$\sum_{w \in N(u)} (1 + h_{w,u}) = 2|E|.$$
2. Let T be a *spanning tree* of G (i.e. T is a tree with vertex set V). Show that there is a *tour* (i.e. a walk with the same starting and ending points) passing each edge of T exactly twice, once for each direction.
3. Let $v_0, v_1, ..., v_{2|V|-2} = v_0$ be the sequence of vertices of such tour. Prove that
$$\sum_{i=0}^{2|V|-3} h_{v_i, v_{i+1}} < 4|V| \times |E|.$$

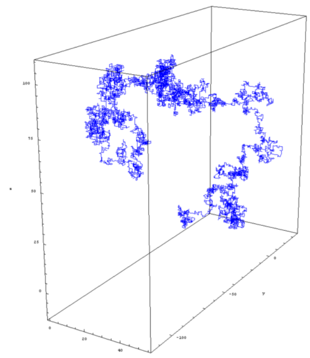
4. Conclude that the cover time of G is upper-bounded by $4|V| \times |E|$.

Exercice 4. *Random walks on $\mathbb{Z}^d$*

Consider the simple random walk over $\mathcal{Z}^d$ with probability $1/2d$ to jump towards any of the $2d$ neighbors in the grid. The walk is clearly irreducible.



A random walk in $\mathcal{Z}^2$
(10000 steps, Wikipedia)



A random walk in $\mathcal{Z}^3$
(10000 steps, Wikipedia)

1. For $d = 1$, whether it is recurrent ? positive recurrent ?
2. For $d = 2$, whether it is recurrent ? positive recurrent ?
   Hint : Consider decomposing the walk into two independent walks.
3. In the case $d = 3$, for every n, show that

$$\mathbb{P}\left(S_{2n} = 0\right) = \sum_{r+s+t=n} \binom{2n}{n} \binom{n}{r,s,t}^2 \frac{1}{6^{2n}},$$

   where S_i is the location of the walk at time i.
4. Show that

$$\sum_{n=0}^{\infty} \mathbb{P}\left(S_{2n} = 0\right) < \infty$$

   and conclude for the case $d = 3$.

Exercice 5. *Cat and mouse*

A cat and mouse each independently take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G.

1. Show an upper bound of $\mathcal{O}(m^2 n)$ on the expected time before the cat eats the mouse. (Hint : Consider a Markov chain whose states are the ordered pairs (a, b), where a is the position of the cat and b is the position of the mouse.)